

THE BRAIDED HEISENBERG GROUP

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ABSTRACT We compute the braided groups and braided matrices $B(R)$ for the solution R of the Yang-Baxter equation associated to the quantum Heisenberg group. We also show that a particular extension of the quantum Heisenberg group is dual to the Heisenberg universal enveloping algebra $U_q(h)$, and use this result to derive an action of $U_q(h)$ on the braided groups. We then demonstrate the various covariance properties using the braided Heisenberg group as an explicit example. In addition, the braided Heisenberg group is found to be self-dual. Finally, we discuss a physical application to a system of n braided harmonic oscillators. An isomorphism is found between the n -fold braided and unbraided tensor products, and the usual ‘free’ time evolution is shown to be equivalent to an action of a primitive generator of $U_q(h)$ on the braided tensor product.

1 Introduction

Braided groups [5, 6, 7] are a variant of quantum groups or superquantum groups in which the ± 1 of super statistics is replaced by a more general braiding, Ψ . This braiding replaces the ordinary twist map, τ , in the Hopf algebra (or bialgebra) axioms. This has its most obvious effect in the tensor product algebra structure, where the usual definition of the product (for quantum groups)

$$(a \otimes b)(c \otimes d) = a\tau(b \otimes c)d = ac \otimes bd \quad (1)$$

must now be rewritten to incorporate the braid statistics

$$(a \otimes b)(c \otimes d) = a\Psi(b \otimes c)d \quad (2)$$

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Note that the braided transposition Ψ may produce a linear combination of terms in the tensor product algebra: this is more general than the modified transposition of superquantum groups, which differs from τ only by a variable sign.

One of the main motivations for introducing braided groups is that they can be used as a tool for doing quantum group calculations in a fully covariant way [6]. For a quantum group, it is possible to define an action of the universal enveloping algebra on itself or the function algebra, which will preserve *either* the product or the coproduct structure, but not both. Braided groups have been modified in precisely such a way as to allow the full Hopf algebra structure to be conserved by the action. The braiding arises as a natural result of this process. All quantum groups have braided group versions [6].

Here we study in detail one example of these braided groups: namely, the braided version of the quantum Heisenberg group. This has been chosen for study as it provides a particularly simple example with which to demonstrate the above properties. It is also of great physical interest, since the universal enveloping algebra $U_q(h)$ is basically the ‘quantum group version’ of the ordinary quantum harmonic oscillator [1, 3]. The algebra of $U_q(h)$ differs from the usual harmonic oscillator algebra only by the fact that the number operator, $N = a^\dagger a$, is viewed as a primitive generator, and the role of Planck’s constant is taken over by a central, grouplike operator (see Section 4 for details.) It is hoped that the close link between the Heisenberg group and such a well-known system may allow some insight to be gained into the physical meaning of some of the mathematical properties: in particular the braiding itself.

We begin in Section 2 by reviewing the explicit structure of $H_q(1)$, the quantum Heisenberg group of function algebra type [1]. We also study two extensions of it which will be useful in later sections, and show how they arise by taking successive quotients of some general 3×3 matrices of Heisenberg type. (The two extensions are, specifically, the upper triangular matrices of Heisenberg type $T_+(h)$, and a central extension $\widetilde{H_q(1)}$.) In Section 3 we go on to compute the corresponding braided groups of function algebra type. The general 3×3 braided matrices

(of Heisenberg type) are calculated first, and then various quotients $BT_+(h)$, $\widetilde{BH_q(1)}$, $BH_q(1)$ are taken, in close analogy to the procedure followed in Section 2.

In Section 4 the covariance properties of both the quantum and braided groups under the action of $U_q(h)$ are discussed. In the process, the duality between $U_q(h)$ and $H_q(1)$ is clarified - it will be shown that it is in fact the central extension $\widetilde{H_q(1)}$ which is dual to $U_q(h)$, not $H_q(1)$ itself. Correspondingly, in the braided case, the centrally extended $\widetilde{BH_q(1)}$ is dual to the braided universal enveloping algebra $BU_q(h)$. In fact, $\widetilde{BH_q(1)}$ is shown to be self-dual, due to the existence of an isomorphism: $\widetilde{BH_q(1)} \cong BU_q(h)$.

We conclude in Section 5 with a physical application to an n-body system of braided harmonic oscillators. We first look for a Fock space representation of the n-fold braided tensor product [5, 6], and in doing so discover an isomorphism between the braided and unbraided tensor product algebras (denoted $BU_q(h)^{\otimes n}$ and $U_q(h)^{\otimes n}$ respectively). The characteristic feature of the braided tensor product is its covariance under the action of $U_q(h)$. We use this action to define a time-evolution of the (braided) system. The isomorphism is then used to map the $U_q(h)$ -action and resulting time-evolution across to the unbraided tensor algebra $U_q(h)^{\otimes n}$, where it is found to correspond to that induced by the free Hamiltonian. Thus it is shown that the usual time evolution of n uncoupled harmonic oscillators, is equivalent to the action of a primitive generator of $U_q(h)$ on the braided harmonic oscillators.

2 The Quantum Heisenberg Group

We begin by recalling the general FRT construction [4] for quantum groups of function algebra type. To any matrix $R \in M_n(\mathbf{C}) \otimes M_n(\mathbf{C})$ which is a solution of the Quantum Yang-Baxter Equation (QYBE), there can be associated a non-commutative algebra generated by 1 and n^2 generators $\{t^i_j\}$, modulo certain relations. When coupled with a standard definition of the coproduct and counit structure, this then forms the quantum (semi)group $A(R)$ associated with R . If T is the $n \times n$ matrix of the generators $\{t^i_j\}$, then the defining relations can be written

as follows in matrix form

$$RT_1T_2 = T_2T_1R \quad (3)$$

$$\Delta(t^i_j) = \sum_k t^i_k \otimes t^k_j, \quad \varepsilon(t^i_j) = \delta^i_j \quad (4)$$

where $T_1 = T \otimes \mathbf{1}$ and $T_2 = \mathbf{1} \otimes T$.

We use the explicit form of the Heisenberg R -matrix given by Celeghini et al. [1]

$$R = \left(\begin{array}{c|ccc|ccc} & & & & & & & \\ & & & & & & & \\ I_3 & 0 & 0 & 0 & 0 & 0 & 0 & \\ & 0 & 0 & \omega & 0 & -\frac{\omega}{2} & 0 & \\ & 0 & 0 & 0 & 0 & 0 & 0 & \\ \hline & 1 & 0 & -\frac{\omega}{2} & & & & \\ 0 & 0 & 1 & 0 & & & 0 & \\ & 0 & 0 & 1 & & & & \\ \hline 0 & & 0 & & & & I_3 & \end{array} \right) \quad (5)$$

The most general quantum group associated with this R is obtained by taking T in Equation (3) to be a general 3×3 matrix (ie. with 9 indeterminates $\{t^i_j\}$). However, we can also consider different quotients of this general group by imposing additional (consistent) relations on the generators $\{t^i_j\}$. The most general quotient we consider explicitly involves what naturally may be called the upper triangular matrices of Heisenberg type. We denote it $T_+(h)$. This is obtained by setting

$$t^i_j = 0, \quad i < j \quad (6)$$

This can be done while maintaining consistency with the FRT construction, since the generators which are being set to zero generate a bi-ideal of $A(R)$. To see this, consider the general form of the coproduct: $\Delta(t^i_j) = t^i_k \otimes t^k_j$. It is evident that if $i < j$, then either $i < k$ or $k < j$, for all k . So $\text{span}\{t^i_j : i < j\}$ will always define a co-ideal of $A(R)$ (for any R .) The upper triangular nature of the Heisenberg R -matrix ensures that setting these generators to zero does not make the relations too trivial.

Using the following notation for remaining matrix of generators

$$T = \begin{pmatrix} d_1 & \alpha & \beta \\ 0 & \gamma & \delta \\ 0 & 0 & d_2 \end{pmatrix} \quad (7)$$

the algebra relations can be written

$$d_1 = d_2 (= d) \quad (8)$$

$$\alpha\beta - \beta\alpha = \frac{\omega}{2}\alpha d, \quad \alpha\delta - \delta\alpha = 0, \quad \beta\delta - \delta\beta = -\frac{\omega}{2}\delta d \quad (9)$$

$$\alpha\gamma - \gamma\alpha = \beta\gamma - \gamma\beta = \delta\gamma - \gamma\delta = d\gamma - \gamma d = 0 \quad (10)$$

$$\alpha d - d\alpha = \beta d - d\beta = \delta d - d\delta = 0 \quad (11)$$

Note that since both d and γ are grouplike, and commute with all other generators, the determinant $D = dd\gamma$ is also central and grouplike, as is its inverse D^{-1} . It is therefore possible to formally adjoin D and D^{-1} to the algebra, with the relations $DD^{-1} = 1 = D^{-1}D$. This then allows the antipode structure to be defined

$$S(\alpha) = -D^{-1}d\alpha, \quad S(\beta) = -D^{-1}\gamma\beta + D^{-1}\alpha\delta, \quad S(\delta) = -D^{-1}d\delta \quad (12)$$

$$S(\gamma) = D^{-1}dd, \quad S(d) = D^{-1}d\gamma, \quad (13)$$

thus giving $T_+(h)$ the full Hopf algebra structure. Note that the FRT construction guarantees that $A(R)$ will always be a bialgebra, but the existence of an antipode is not necessarily assured.

A further quotient can be taken by setting the diagonal elements of T to 1.

$$t^i_i = 1 \quad (14)$$

Note that this can be done while retaining the Hopf algebra structure, since γ and d are both central and grouplike, and can therefore be set to 1 without invalidating any of the axioms. The algebra and antipode relations then reduce to

$$\alpha\beta - \beta\alpha = \frac{\omega}{2}\alpha, \quad \alpha\delta - \delta\alpha = 0, \quad \beta\delta - \delta\beta = -\frac{\omega}{2}\delta \quad (15)$$

$$S(\alpha) = -\alpha, \quad S(\beta) = -\beta = \alpha\delta, \quad S(\delta) = -\delta \quad (16)$$

It is this quotient which is commonly known [1] as the quantum Heisenberg group, $H_q(1)$.

Valid Hopf algebras could also be obtained by taking quotients intermediate between $T_+(h)$ and $H_q(1)$; ie. by setting some, but not all, of the diagonal elements to 1. Of particular interest

to us is the case where d is set to 1, but γ is retained. As we shall see in a later section, this algebra turns out to be dual to the enveloping Heisenberg algebra. We shall call it the extended quantum Heisenberg group, and denote it $\widetilde{H_q(1)}$. Explicitly it looks like

$$\alpha\beta - \beta\alpha = \frac{\omega}{2}\alpha, \quad \alpha\delta - \delta\alpha = 0, \quad \beta\delta - \delta\beta = -\frac{\omega}{2}\delta \quad (17)$$

$$\alpha\gamma - \gamma\alpha = \beta\gamma - \gamma\beta = \delta\gamma - \gamma\delta = 0 \quad (18)$$

$$S(\alpha) = -\gamma^{-1}\alpha, \quad S(\beta) = -\beta + \gamma^{-1}\alpha\delta, \quad S(\delta) = -\gamma^{-1}\delta, \quad S(\gamma) = \gamma^{-1} \quad (19)$$

where γ^{-1} , with relations $\gamma\gamma^{-1} = 1 = \gamma^{-1}\gamma$, can be formally adjoined to the algebra in a similar way to D and D^{-1} for $T_+(h)$, to allow convenient expression of the antipode.

3 The Braided Heisenberg group

The general construction for $B(R)$ [6], the braided matrices (or, after quotienting, the braided group) associated with R , is similar to that for $A(R)$. It consists again of n^2 generators u^i_j constrained by matrix relations. The definition of the coproduct and counit structure is also again of matrix type

$$\underline{\Delta}u^i_j = u^i_k \otimes u^k_j, \quad \underline{\varepsilon}(u^i_j) = \delta^i_j \quad (20)$$

As was pointed out in the Introduction, the tensor product algebra structure is quite different to that for quantum groups, due to the inclusion of the braided transposition. This necessitates a modification of the product on the generators; that is, $B(R)$ will in general be different as an algebra to the corresponding $A(R)$. The defining algebra relations and braiding are as given in [6, 7] in matrix form, using the same notation as in Section 2.

$$R_{21}U_1R_{12}U_2 = U_2R_{21}U_1R_{12} \quad (21)$$

$$\Psi(u^I \otimes u^K) = u^L \otimes u^J \Psi^K{}^I{}_L{}_J \quad (22)$$

where I, J , etc. are multi-indices (i_0, i_1) for the components $u^{i_0}{}_{i_1}$, $R_{12} = R$, $R_{21} = \tau(R_{12})$. Equation (21) is also known in various other contexts. $\Psi^K{}^I{}_L{}_J$ is defined as follows in terms of

R

$$\Psi_{(j_0, j_1)}^{(i_0, i_1)}{}_{(l_0, l_1)}^{(k_0, k_1)} = R_{a \ j_0}^{k_0 \ d} R_{l_0 \ j_1}^{-1 \ a} R_{b \ i_1}^{l_1 \ c} \tilde{R}_{k_1 \ i_0}^c{}^d \quad (23)$$

where $\tilde{R} = ((R^{t_2})^{-1})^{t_2}$, with t_2 denoting transpose in the second factor.

We compute the braided matrices of Heisenberg type, $BM_h(3)$, defined as $B(R)$ for the Heisenberg R -matrix shown in Equation (5). The computer program REDUCE was used for this calculation. We find that considerable symmetry is apparent in the algebra relations thus obtained, even for these general matrices before any quotients are taken, and it may therefore be of interest to record them in full. Firstly, it is found that $u^3{}_1$ commutes with all other generators

$$u^3{}_1 u^i{}_j - u^i{}_j u^3{}_1 = 0 \quad (24)$$

The diagonal elements and $u^1{}_3$ commute among themselves

$$u^i{}_i u^j{}_j - u^j{}_j u^i{}_i = 0 \quad (25)$$

$$u^i{}_i u^1{}_3 - u^1{}_3 u^i{}_i = 0 \quad (26)$$

and a certain symmetry can be observed in their commutators with the remaining generators

$$u^i{}_i u^j{}_k - u^j{}_k u^i{}_i = (-1)^{i+k} c_i \omega u^3{}_1 u^j{}_k \quad c_i = \begin{cases} \frac{1}{2} & i = 1, 3 \\ 1 & i = 2 \end{cases} \quad (27)$$

$$u^1{}_3 u^j{}_k - u^j{}_k u^1{}_3 = \frac{\omega^2}{4} u^3{}_1 u^j{}_k + (-1)^j u^j{}_k (u^1{}_1 + u^3{}_3) \quad (28)$$

where $u^j{}_k = u^1{}_2, u^2{}_1, u^2{}_3, u^3{}_2$. The relations of these generators among themselves are

$$u^1{}_2 u^3{}_2 - u^3{}_2 u^1{}_2 = -\frac{\omega}{2} (u^3{}_2)^2 \quad (29)$$

$$u^2{}_1 u^2{}_3 - u^2{}_3 u^2{}_1 = -\frac{\omega}{2} (u^2{}_1)^2 \quad (30)$$

$$u^2{}_1 u^2{}_3 - u^2{}_3 u^2{}_1 = \omega (u^3{}_1)^2 \quad (31)$$

$$u^1{}_2 u^2{}_1 - u^2{}_1 u^1{}_2 = \frac{\omega}{2} u^3{}_1 (u^2{}_2 - 2u^1{}_1) \quad (32)$$

$$u^3{}_2 u^2{}_3 - u^2{}_3 u^3{}_2 = \frac{\omega}{2} u^3{}_1 (u^2{}_2 - 2u^3{}_3) \quad (33)$$

$$u^1_2 u^2_3 - u^2_3 u^1_2 = -\frac{\omega^2}{2} u^3_1 (u^1_1 - u^2_2) + \omega u^1_1 (u^2_2 - u^3_3) + \frac{\omega}{2} (u^2_3 u^3_2 - u^1_2 u^2_1) \quad (34)$$

Also, some additional relations are imposed by the fact that the defining matrix equation gives two different values for some commutators, which must be set equal

$$u^3_1 u^1_2 = u^1_1 u^3_2, \quad u^3_1 u^2_2 = u^2_1 u^3_2, \quad u^3_1 u^2_3 = u^2_1 u^3_3 \quad (35)$$

$$u^2_3 u^3_2 - u^2_1 u^1_2 = u^2_2 (u^3_3 - u^1_1) \quad (36)$$

Lemma 3.1 *The sub-diagonal elements of U , ie. the generators u^2_1 , u^3_1 and u^3_2 , can be set to zero in a manner consistent with both the algebra relations and the braiding.*

Proof We first note that consistency with the algebra and braiding is all that needs to be checked, since, as was pointed out in Section 2, setting below-diagonal elements to zero will always be consistent with a coproduct structure of matrix type. It can be seen from the algebra relations that all commutators involving any of these three generators have right hand sides which also go to zero in this quotient. Of the four remaining relations, three are empty, and the other gives the condition

$$u^2_2 (u^3_3 - u^1_1) = 0 \quad (37)$$

If this condition is satisfied, then the quotient can be seen to be consistent with the algebra relations. Since it is obviously desirable to retain u^2_2 in the algebra, we shall set

$$u^1_1 = u^3_3 = d \quad (38)$$

Consistency with the braiding must now be checked. Since there are 81 braid relations, we will not show them explicitly, but merely note that all braided transpositions of tensor products involving u^2_1, u^3_1 or u^3_2 contain at least one of these generators in all terms on the right hand side of the equation. So, having imposed the condition (38), no inconsistencies arise on taking the desired quotient. \square

We shall call the quotient described by Lemma (3.1) the ‘braided upper triangular matrices of Heisenberg type’, and denote it $BT_+(h)$. To show its structure explicitly, it will be convenient

to adopt the same notation as was defined in Equation (7) for the quantum group $T_+(h)$. The first thing to note, then, is that the diagonal elements d and γ are bosonic generators in the sense that

$$\Psi(f \otimes \gamma) = \gamma \otimes f, \quad \Psi(\gamma \otimes f) = f \otimes \gamma, \quad \Psi(f \otimes d) = d \otimes f, \quad \Psi(d \otimes f) = f \otimes d \quad \forall f. \quad (39)$$

These elements are also central

$$\gamma f = f \gamma, \quad df = fd \quad \forall f. \quad (40)$$

The other algebra relations are

$$\alpha\beta - \beta\alpha = \omega d\alpha, \quad \alpha\delta - \delta\alpha = \omega d(\gamma - d), \quad \beta\delta - \delta\beta = \omega d\delta \quad (41)$$

and the rest of the braiding is given by

$$\begin{aligned} \Psi(\alpha \otimes \alpha) &= \alpha \otimes \alpha & \Psi(\beta \otimes \delta) &= \delta \otimes \beta + \omega(d - \gamma) \otimes \delta \\ \Psi(\alpha \otimes \beta) &= \beta \otimes \alpha + \omega \alpha \otimes (d - \gamma) & \Psi(\delta \otimes \alpha) &= \alpha \otimes \delta \\ \Psi(\alpha \otimes \delta) &= \delta \otimes \alpha + \omega(d - \gamma) \otimes (\gamma - d) & \Psi(\delta \otimes \beta) &= \beta \otimes \delta \\ \Psi(\beta \otimes \alpha) &= \alpha \otimes \beta & \Psi(\delta \otimes \delta) &= \delta \otimes \delta \\ \Psi(\beta \otimes \beta) &= \beta \otimes \beta + \omega \alpha \otimes \delta \end{aligned} \quad (42)$$

Proposition 3.2 *The braided matrices $BT_+(h)$ have a braided determinant $D = dd\gamma$ which can be adjoined to the algebra, as can also its inverse D^{-1} . The braided antipode on $BT_+(h)$ is then given by:*

$$\underline{S}(\alpha) = -D^{-1}d\alpha, \quad \underline{S}(\beta) = -D^{-1}\gamma\beta + D^{-1}\alpha\delta, \quad \underline{S}(\delta) = -D^{-1}d\delta$$

$$\underline{S}(\gamma) = D^{-1}dd, \quad \underline{S}(d) = D^{-1}d\gamma.$$

Proof Since the elements d and γ are both central, grouplike and bosonic, D and D^{-1} will also have these properties. They can therefore be formally adjoined to the algebra, with relations $DD^{-1} = 1 = D^{-1}D$. That the antipode is as given by Proposition (3.2) can be verified by direct substitution into the defining axiom

$$.(\underline{S} \otimes id) \circ \underline{\Delta} a = \underline{\varepsilon} a. \quad (43)$$

□

Note that it is special to the present example that the antipode on the braided group $BT_+(H)$ is the same on the generators as that of the quantum group $T_+(h)$. Such similarity is not generally observed between quantum groups and their braided counterparts. The similarity does *not* however, extend to higher order products of the generators. The antipode on $BT_+(h)$ is truly braided in the sense that

$$\underline{S}(ab) = .(\underline{S} \otimes \underline{S}) \circ \Psi(a \otimes b) \quad (44)$$

whereas for a quantum group the antipode extends as a straightforward anti-algebra map

$$S(ab) = .(S \otimes S) \circ \tau(a \otimes b) = S(b)S(a) \quad (45)$$

From Equations (39) and (40) it is evident that the further quotient

$$\gamma = d = 1 \quad (46)$$

can be taken, to obtain the braided version of the usual quantum Heisenberg group, which we shall call $BH_q(1)$. Its algebra relations, braiding and antipode are given by

$$\alpha\beta - \beta\alpha = \omega\alpha, \quad \alpha\delta - \delta\alpha = 0, \quad \beta\delta - \delta\beta = \omega\delta \quad (47)$$

$$\begin{aligned} \Psi(\alpha \otimes \alpha) &= \alpha \otimes \alpha & \Psi(\beta \otimes \delta) &= \delta \otimes \beta \\ \Psi(\alpha \otimes \beta) &= \beta \otimes \alpha & \Psi(\delta \otimes \alpha) &= \alpha \otimes \delta \\ \Psi(\alpha \otimes \delta) &= \delta \otimes \alpha & \Psi(\delta \otimes \beta) &= \beta \otimes \delta \\ \Psi(\beta \otimes \alpha) &= \alpha \otimes \beta & \Psi(\delta \otimes \delta) &= \delta \otimes \delta \\ \Psi(\beta \otimes \beta) &= \beta \otimes \beta + \omega\alpha \otimes \delta \end{aligned} \quad (48)$$

$$\underline{S}(\alpha) = -\alpha, \quad \underline{S}(\beta) = -\beta = \alpha\delta, \quad \underline{S}(\delta) = -\delta \quad (49)$$

A comparison of Equations (47), (48) and (49) with (15) and (16) reveals that this algebra is remarkably similar to that of the quantum Heisenberg group, $H_q(1)$. Also the braiding, while taking an extremely simple form, remains non-trivial in this quotient, ie. $\Psi^2 \neq id$. $BH_q(1)$, then, is probably the simplest example available of a non-trivial braided group.

The braided Heisenberg group can be extended in the same way as the quantum group by retaining γ and setting only d to 1. Let us denote this ‘extended Heisenberg group’ by $\widetilde{BH_q(1)}$.

It has the explicit algebra structure

$$\alpha\beta - \beta\alpha = \omega\alpha, \quad \alpha\delta - \delta\alpha = \omega(\gamma - 1), \quad \beta\delta - \delta\beta = \omega\delta. \quad (50)$$

The braided antipode is

$$\underline{S}(\alpha) = -\gamma^{-1}\alpha, \quad \underline{S}(\beta) = -\beta + \gamma^{-1}\alpha\delta, \quad \underline{S}(\delta) = -\gamma^{-1}\delta, \quad \underline{S}(\gamma) = \gamma^{-1} \quad (51)$$

where γ^{-1} has been formally adjoined to the algebra as a central, grouplike and bosonic element, with relations $\gamma\gamma^{-1} = 1 = \gamma^{-1}\gamma$. Since γ is bosonic, its braiding with the other generators is just simple transposition, as given by Equation (39). The rest of the braiding can easily be obtained from Equation (42), by setting d to 1. Thus, as a bialgebra, $\widetilde{BH_q}(1)$ differs from $BH_q(1)$ only by the addition of this central, grouplike and bosonic element γ , which appears on the right in Equation (50), and in one of the braiding relations. This extension, however, plays an important role in the next section.

4 Braided Heisenberg groups as $U_q(h)$ -modules

In this section we show that $BM_h(3)$, $BT_+(h)$, $\widetilde{BH_q}(1)$ and $BH_q(1)$ are all objects in the same braided category, namely, the category of the representations of $U_q(h)$, the quantum enveloping Heisenberg algebra. This follows from the general theory in [5, 6], but we feel it is of interest to see it explicitly for the present case.

To demonstrate this we first need the explicit structure of the quantum group $U_q(h)$, as given by [1, 3]

$$[A, A^\dagger] = \frac{1}{\omega}(e^{\omega H/2} - e^{-\omega H/2}) \quad [N, A] = -A \quad [N, A^\dagger] = A^\dagger \quad H \text{ central} \quad (52)$$

$$\begin{aligned} \Delta A &= e^{-\omega H/4} \otimes A + A \otimes e^{\omega H/4} \\ \Delta A^\dagger &= e^{-\omega H/4} \otimes A^\dagger + A^\dagger \otimes e^{\omega H/4} \\ \Delta N &= N \otimes 1 + 1 \otimes N \\ \Delta H &= H \otimes 1 + 1 \otimes H \end{aligned} \quad (53)$$

$$\varepsilon X = 0 \quad SX = -X \quad X = A, A^\dagger, N, H. \quad (54)$$

The quasitriangular structure [2], or universal R -matrix, is given by

$$\mathcal{R} = e^{-\omega/2(H \otimes N + N \otimes H)} e^{\omega(e^{\omega H/4} A \otimes e^{-\omega H/4} A^\dagger)} \quad (55)$$

The notation above is the same as that used by [1], but we differ from them in taking N to be a primitive generator (they do note this as a possibility), as was done by [3].

The statement that $BM_h(3)$, etc. are all objects in the category of representations of this quantum group means that $U_q(h)$ acts on all of these braided groups, and that this action is respected by all maps within each group (ie. all the braided Hopf algebra operations $(\underline{\Delta}, \underline{\varepsilon}, \underline{S}, \underline{\varepsilon})$ are covariant under this action.) It is worth noting that the quantum groups of function algebra type $(T_+(h), \widetilde{H_q(1)}$ and $H_q(1)$) are $U_q(h)$ -module coalgebras in this sense, under the quantum coadjoint action. Their products, however, do not respect this action. It is the modified product of the braided versions which makes them into $U_q(h)$ -module Hopf algebras, as explained in general in [8].

Proposition 4.1 $\widetilde{H_q(1)}$ is paired to $U_q(h)$.

Proof To show that $U_q(h)$ and $\widetilde{H_q(1)}$ are paired, it is necessary to check that the following pairing relations

$$\langle \phi\psi, a \rangle = \langle \phi \otimes \psi, \Delta a \rangle \quad (56)$$

$$\langle 1, a \rangle = \varepsilon(a) \quad (57)$$

$$\langle \Delta\phi, a \otimes b \rangle = \langle \phi, ab \rangle \quad (58)$$

$$\varepsilon(\phi) = \langle \phi, 1 \rangle \quad (59)$$

$$\langle S\phi, a \rangle = \langle \phi, Sa \rangle \quad (60)$$

are satisfied, where $\phi, \psi \in U_q(h)$, $a, b \in H_q(1)$. We define the pairing using the matrix representation of $U_q(h)$ given by [1]

$$\rho(A) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \rho(A^\dagger) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\rho(H) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \rho(N) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (61)$$

taking

$$\langle X, t^i_j \rangle = \rho(X)^i_j \quad X = A, A^\dagger, H, N. \quad (62)$$

The relation (57) can then immediately be seen to hold on the generators

$$\langle 1, t^i_j \rangle = I_3^i_j = \delta^i_j = \varepsilon(t^i_j). \quad (63)$$

Also,

$$\langle X, 1 \rangle = \langle X, t^1_1 \rangle = \langle X, t^3_3 \rangle = 0 = \varepsilon X \quad (64)$$

so that (59) is also seen to hold. Note that it is this relation which would fail to be satisfied were γ not retained, since

$$\langle N, t^2_2 \rangle = 1 \neq 0. \quad (65)$$

The relations (56) and (58) can be used to define the extension of the pairing to higher order products of the generators, according to

$$\langle XY, t^i_j \rangle = \rho(XY)^i_j = \rho(X)^i_k \rho(X)^k_j \quad (66)$$

$$\langle X, t^i_j t^k_l \rangle = \langle X_{(1)}, t^i_j \rangle \langle X_{(2)}, t^k_l \rangle = \rho(X_{(1)})^i_j \rho(X_{(2)})^k_l \quad (67)$$

where $\Delta X = X_{(1)} \otimes X_{(2)}$. The consistency of these definitions with the algebra relations of $U_q(h)$ and $\widetilde{H_q(1)}$ respectively must then be checked. It can easily be verified that the definition (66) respects the algebra (52) simply by multiplying out the matrices (61). The consistency of the second definition is most easily shown by a direct computation of all such pairings. The relevant non-zero cases are

$$\begin{aligned} \langle A, \alpha\beta \rangle &= \frac{\omega}{4} & \langle A, \beta\alpha \rangle &= -\frac{\omega}{4} \\ \langle A^\dagger, \beta\delta \rangle &= -\frac{\omega}{4} & \langle A^\dagger, \delta\beta \rangle &= \frac{\omega}{4} \\ \langle A, \alpha\gamma \rangle &= \langle A, \gamma\alpha \rangle = \langle A, \alpha\gamma^{-1} \rangle = \langle A, \gamma^{-1}\alpha \rangle = 1 \end{aligned}$$

$$\begin{aligned}
\langle A^\dagger, \delta\gamma \rangle &= \langle A^\dagger, \gamma\delta \rangle = \langle A^\dagger, \delta\gamma^{-1} \rangle = \langle A^\dagger, \gamma^{-1}\delta \rangle = 1 \\
\langle H, \beta\gamma \rangle &= \langle H, \gamma\beta \rangle = \langle H, \beta\gamma^{-1} \rangle = \langle H, \gamma^{-1}\beta \rangle = 1
\end{aligned} \tag{68}$$

Consistency with the algebra relations (50) is then easily checked. To check the last pairing relation, it is first necessary to define the pairing between the generators of $U_q(h)$ and γ^{-1} . This can be done using the inverse property $\gamma\gamma^{-1} = 1 = \gamma^{-1}\gamma$, and equation (67)

$$\langle X, 1 \rangle = \langle X, \gamma^{-1}\gamma \rangle = \sum \langle X_{(1)}, \gamma^{-1} \rangle \rho(X_{(2)})^2 \quad X = A, A^\dagger, H, N, 1. \tag{69}$$

where $\Delta X = \sum X_{(1)} \otimes X_{(2)}$. The pairings $\langle X_{(1)}, \gamma^{-1} \rangle$ are the only unknowns in this set of equations, and are thus determined by them. They are, explicitly

$$\begin{aligned}
\langle A, \gamma^{-1} \rangle &= 0 & \langle A^\dagger, \gamma^{-1} \rangle &= 0 & \langle N, \gamma^{-1} \rangle &= -1 \\
\langle H, \gamma^{-1} \rangle &= 0 & \langle 1, \gamma^{-1} \rangle &= 1
\end{aligned} \tag{70}$$

The relation (60) can then easily be checked directly, case by case, and is found to hold. Thus it is established that $U_q(h)$ and $\widetilde{H_q(1)}$ are paired. \square

By inspection, the pairing between $U_q(h)$ and $\widetilde{H_q(1)}$ can be seen to be non-degenerate (no generator pairs as zero with all other generators on the other side). Note that all quantum groups of function algebra type $BM_h(3)$ and $BT_+(h)$ are also paired to $U_q(h)$, although in these cases the pairing is degenerate even at generator level. $H_q(1)$ is not paired at all, due to the failure of relation (59), as previously pointed out.

Corollary 4.2 *The coadjoint action of $U_q(h)$ on $B\widetilde{H_q(1)}$ is*

$$\begin{aligned}
A \triangleright \begin{pmatrix} 1 & \alpha & \beta \\ 0 & \gamma & \delta \\ 0 & 0 & 1 \end{pmatrix} &= \begin{pmatrix} 0 & 1-\gamma & -\delta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
A^\dagger \triangleright \begin{pmatrix} 1 & \alpha & \beta \\ 0 & \gamma & \delta \\ 0 & 0 & 1 \end{pmatrix} &= \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & \gamma-1 \\ 0 & 0 & 0 \end{pmatrix} \\
N \triangleright \begin{pmatrix} 1 & \alpha & \beta \\ 0 & \gamma & \delta \\ 0 & 0 & 1 \end{pmatrix} &= \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & -\delta \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

$$H \triangleright \begin{pmatrix} 1 & \alpha & \beta \\ 0 & \gamma & \delta \\ 0 & 0 & 1 \end{pmatrix} = 0 \quad (71)$$

Proof The above duality of $U_q(h)$ and $\widetilde{H_q(1)}$ allows the coadjoint action of $U_q(h)$ on $\widetilde{H_q(1)}$ to be defined in terms of the pairing

$$X \triangleright t^i{}_j = t^k{}_l \langle X, (St^i{}_k) t^l{}_j \rangle. \quad (72)$$

According to theory described in [6], the braided group of function algebra type is obtained from the quantum group of the same type, in the quotient dual to the enveloping algebra, via a process of “transmutation”. A feature of this theory is that the generators of the braided group obtained in this way, transform in the same way as the generators of the quantum group, under the action of the dual enveloping algebra. In the case of the Heisenberg group, this means that the same set of coefficients ($\langle X, (St^i{}_k) t^l{}_j \rangle$) give the action of $U_q(h)$ on $B\widetilde{H_q(1)}$, as give its action on $\widetilde{H_q(1)}$

$$X \triangleright u^i{}_j = u^k{}_l \langle X, (St^i{}_k) t^l{}_j \rangle. \quad (73)$$

A straightforward application of this formula yields the actions quoted. \square

Since a key motivation for introducing braided groups was their covariance properties, it may be of interest at this stage to see explicitly why $B\widetilde{H_q(1)}$ is $U_q(h)$ -covariant, while $\widetilde{H_q(1)}$ is not. Since the action chosen was the quantum coadjoint action, it will by definition respect the coproduct structure of both groups. As previously pointed out, it is the product of $\widetilde{H_q(1)}$ which we expect not to be covariant under this action. It is sufficient to demonstrate this with an explicit example. Thus

$$\begin{aligned} A^\dagger \triangleright (\beta\delta - \delta\beta) &= A^\dagger \triangleright (\beta\delta) - A^\dagger \triangleright (\delta\beta) \\ &= (A^\dagger{}_{(1)} \triangleright \beta)(A^\dagger{}_{(2)} \triangleright \delta) - (A^\dagger{}_{(1)} \triangleright \delta)(A^\dagger{}_{(2)} \triangleright \beta) \\ &= (e^{-\frac{\omega H}{4}} \triangleright \beta)(A^\dagger \triangleright \delta) + (A^\dagger \triangleright \beta)(e^{\frac{\omega H}{4}} \triangleright \delta) \\ &\quad - (e^{-\frac{\omega H}{4}} \triangleright \delta)(A^\dagger \triangleright \beta) - (A^\dagger \triangleright \delta)(e^{\frac{\omega H}{4}} \triangleright \beta) \end{aligned}$$

$$\begin{aligned}
&= \beta(\gamma - 1) + \alpha\delta - \delta\alpha - (\gamma - 1)\beta \\
&= \alpha\delta - \delta\alpha \\
&= \begin{cases} 0 & \widetilde{H_q(1)} \\ \omega(\gamma - 1) & \widetilde{BH_q(1)} \end{cases} \quad (74)
\end{aligned}$$

Now,

$$\beta\delta - \delta\beta = \begin{cases} -\frac{\omega}{2}\delta & \widetilde{H_q(1)} \\ \omega\delta & \widetilde{BH_q(1)} \end{cases} \quad (75)$$

so that for $\widetilde{BH_q(1)}$

$$\begin{aligned}
A^\dagger \triangleright (\omega\delta) &= \omega(\gamma - 1) \\
&= A^\dagger \triangleright (\beta\delta - \delta\beta) \quad (76)
\end{aligned}$$

while for $\widetilde{H_q(1)}$

$$\begin{aligned}
A^\dagger \triangleright (-\frac{\omega}{2}\delta) &= -\frac{\omega}{2}(\gamma - 1) \\
&\neq A^\dagger \triangleright (\beta\delta - \delta\beta) \quad (77)
\end{aligned}$$

It can be seen that while the action of A^\dagger preserves the commutator in $\widetilde{BH_q(1)}$, this is not the case for $\widetilde{H_q(1)}$, where the action of A^\dagger on the two halves of the commutator give different results: the product is not respected.

Proceeding in similar fashion, it can easily be shown that all commutators of $\widetilde{BH_q(1)}$ are preserved by the action given in Corollary 4.2. By contrast, one other action on $\widetilde{H_q(1)}$ will be found to fail: that of A on the commutator $[\alpha, \beta]$.

In fact formula (73) works more generally: it is true for any braided group which maps to $\widetilde{BH_q(1)}$ (eg. $BM(3)$, $BT_+(h)$). According to general theory [8], any universal enveloping algebra can be defined as being generated by 1 and $2n^2$ indeterminates ($2n \times n$ matrices l^+ , l^-), modulo certain matrix relations. For standard R-matrices, this is part of the FRT [4] approach, but it is also valid in the general case. Following [7] we first note that these generators can be expressed in terms of the universal R -matrix

$$l^+ = \mathcal{R}^{(1)} \langle t, \mathcal{R}^{(2)} \rangle, \quad l^- = \langle t, S\mathcal{R}^{(1)} \rangle \mathcal{R}^{(2)} \quad (78)$$

where $\mathcal{R} = \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}$. The action of the universal enveloping algebra on $B(R)$ is then given in [6, 7]

$$l^{+k}{}_l \triangleright u^i{}_j = R^{-1i}{}_m{}^k{}_a u^m{}_n R^n{}_j{}^a{}_l \quad l^{-i}{}_j \triangleright u^k{}_l = R^i{}_a{}^k{}_m u^m{}_n R^{-1a}{}_j{}^n{}_l \quad (79)$$

Careful comparison of the notation will show that this is, in fact, equivalent to Equation (73), (using some of the properties of the pairing.)

For $U_q(h)$, l^+ and l^- can be computed from (78), and are found to be, explicitly

$$l^+ = \begin{pmatrix} 1 & 0 & -\frac{\omega}{2}N \\ 0 & e^{-\frac{\omega H}{2}} & \omega e^{-\frac{\omega H}{4}}A \\ 0 & 0 & 1 \end{pmatrix} \quad l^- = \begin{pmatrix} 1 & -\omega e^{-\frac{\omega H}{4}}A^\dagger & \frac{\omega}{2}N \\ 0 & e^{\frac{\omega H}{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (80)$$

Using Equation (79), we then compute the general action of $U_q(h)$ on $BM_h(3)$. In terms of the generators, it is

$$\begin{aligned} A \triangleright u &= \begin{pmatrix} -u_{21} & u_{11} - u_{22} & -u_{23} - \frac{\omega}{2}u_{21} \\ 0 & u_{21} & 0 \\ 0 & u_{31} & 0 \end{pmatrix} \\ A^\dagger \triangleright u &= \begin{pmatrix} 0 & 0 & u_{12} + \frac{\omega}{2}u_{32} \\ -u_{31} & -u_{32} & u_{22} - u_{33} \\ 0 & 0 & u_{32} \end{pmatrix} \\ N \triangleright u &= \begin{pmatrix} 0 & u_{12} & 0 \\ -u_{21} & 0 & -u_{23} \\ 0 & u_{32} & 0 \end{pmatrix} \\ e^{\pm \frac{\omega H}{2}} \triangleright u &= \begin{pmatrix} u_{11} \mp \frac{\omega}{2}u_{31} & u_{12} \mp u_{32} & u_{13} \mp \frac{\omega}{2}(u_{33} - u_{11}) - \frac{\omega^2}{4}u_{31} \\ u_{21} & u_{22} & u_{23} \pm \frac{\omega}{2}u_{21} \\ u_{31} & u_{32} & u_{33} \pm \frac{\omega}{2}u_{31} \end{pmatrix} \end{aligned} \quad (81)$$

It is easy to see, by inspection, that these actions reduce to those already given for $B\widetilde{H}_q(1)$, when the relevant quotient is taken.

All of the $U_q(h)$ -modules which have thus far been considered have been braided groups of function algebra type. It is also possible for $U_q(h)$ to act on quantum or braided groups of enveloping algebra type. $U_q(h)$ does in fact act upon itself, as an algebra only, via the quantum adjoint action. This action does not, however, respect its coalgebra structure. In analogy with the function algebra case, a braided version of $U_q(h)$ can be defined which has the same algebra,

but a modified coproduct, which will then be covariant under the action [5]. It may naturally be denoted $BU_q(h)$. See [5] for details of the construction.

Theorem 4.3 $\widetilde{BH}_q(1)$ is isomorphic to $BU_q(h)$.

Proof Let the matrix L be defined by

$$L = l^+ S l^- \quad (82)$$

Then the identification $u = L$ always gives a homomorphism between the braided groups of function and enveloping algebra types [7]. In this case, the map $u \mapsto L$ will give a homomorphism $\widetilde{BH}_q(1) \mapsto BU_q(h)$. For the Heisenberg group, L is, explicitly

$$L = \begin{pmatrix} 1 & \omega e^{-\frac{\omega H}{4}} A^\dagger & -\omega N \\ 0 & e^{-\omega H} & \omega e^{-\frac{\omega H}{4}} A \\ 0 & 0 & 1 \end{pmatrix} \quad (83)$$

It can easily be verified directly that this does indeed give an algebra homomorphism, (remembering that by definition $BU_q(h)$ has the same algebra (on the generators) as $U_q(h)$.) Also, the map $u \mapsto L$ is obviously one-to-one when u is taken to be the matrix of generators in the quotient $\widetilde{BH}_q(1)$. Thus the identifications

$$\alpha = \omega e^{-\frac{\omega H}{4}} A^\dagger, \quad \beta = -\omega N, \quad \gamma = e^{-\omega H}, \quad \delta = \omega e^{-\frac{\omega H}{4}} A \quad (84)$$

define an isomorphism between $\widetilde{BH}_q(1)$ and $BU_q(h)$. \square

This isomorphism can be exploited to derive the coalgebra structure (etc.) of $BU_q(h)$ directly from that of $\widetilde{BH}_q(1)$. The coalgebra and antipode are

$$\begin{aligned} \underline{\Delta} A &= A \otimes e^{\frac{\omega H}{4}} + e^{-3\frac{\omega H}{4}} \otimes A \\ \underline{\Delta} A^\dagger &= A^\dagger \otimes e^{-3\frac{\omega H}{4}} + e^{\frac{\omega H}{4}} \otimes A^\dagger \\ \underline{\Delta} N &= N \otimes 1 + 1 \otimes N - \omega e^{-\frac{\omega H}{4}} A^\dagger \otimes e^{-\frac{\omega H}{4}} A \\ \underline{\Delta} H &= H \otimes 1 + 1 \otimes H \end{aligned} \quad (85)$$

$$\underline{S}(A) = -e^{\frac{\omega H}{2}} A, \quad \underline{S}(A^\dagger) = -e^{\frac{\omega H}{2}} A^\dagger, \quad \underline{S}(N) = N - e^{\frac{\omega H}{2}} A^\dagger A, \quad \underline{S}(H) = -H. \quad (86)$$

The coproduct on $BU_q(h)$ is seen to be considerably less symmetric than that on $U_q(h)$. A more complicated coalgebra structure is often observed in braided universal enveloping algebras, as compared to their quantum counterparts. This can be partially accounted for by the fact that the universal R -matrix is just 1 for the braided group, and some of the information which used to be contained (in the quantum group) in this quasi-triangular structure, is now “transferred” to the coalgebra structure. In $BU_q(h)$, for example, a similarity can be observed between the extra term in the coproduct of N and part of the R -matrix of $U_q(h)$.

The action of $U_q(h)$ on $BU_q(h)$ can also be derived directly from the isomorphism. It is in fact the same as the adjoint action of $U_q(h)$ on itself, as defined by the formula [8, Section 6]

$$h \triangleright a = h_{(1)} a S h_{(2)} \quad (87)$$

where $\Delta h = h_{(1)} \otimes h_{(2)}$. Explicitly

$$\begin{aligned} A \triangleright \begin{pmatrix} A^\dagger & N \\ H & A \end{pmatrix} &= \begin{pmatrix} e^{-\frac{\omega H}{4}}[H] & e^{-\frac{\omega H}{4}}A \\ 0 & 0 \end{pmatrix} \\ A^\dagger \triangleright \begin{pmatrix} A^\dagger & N \\ H & A \end{pmatrix} &= \begin{pmatrix} 0 & -e^{-\frac{\omega H}{4}}A^\dagger \\ 0 & -e^{-\frac{\omega H}{4}}[H] \end{pmatrix} \\ N \triangleright \begin{pmatrix} A^\dagger & N \\ H & A \end{pmatrix} &= \begin{pmatrix} A^\dagger & 0 \\ 0 & -A \end{pmatrix} \\ H \triangleright \begin{pmatrix} A^\dagger & N \\ H & A \end{pmatrix} &= 0 \end{aligned} \quad (88)$$

The quantum adjoint action is always respected by the product. It is also respected by the modified coproduct of $BU_q(h)$ (since it is isomorphic to the coadjoint action on the function algebra.) Thus $BU_q(h)$ is $U_q(h)$ -covariant.

To complete the description of $BU_q(h)$, it only remains to give the braiding. This can be computed using the isomorphism in Theorem 4.3.

$$\Psi(A^\dagger \otimes A^\dagger) = A^\dagger \otimes A^\dagger$$

$$\begin{aligned}
\Psi(A^\dagger \otimes N) &= N \otimes A^\dagger - \omega e^{-\frac{\omega H}{4}} A^\dagger \otimes e^{-\frac{\omega H}{4}} [H] \\
\Psi(A^\dagger \otimes A) &= A \otimes A^\dagger - \omega e^{-\frac{\omega H}{4}} [H] \otimes e^{-\frac{\omega H}{4}} [H] \\
\Psi(N \otimes A^\dagger) &= A^\dagger \otimes N \\
\Psi(N \otimes N) &= N \otimes N - \omega e^{-\frac{\omega H}{4}} A^\dagger \otimes e^{-\frac{\omega H}{4}} A \\
\Psi(N \otimes A) &= A \otimes N - \omega e^{-\frac{\omega H}{4}} [H] \otimes e^{-\frac{\omega H}{4}} A \\
\Psi(A \otimes A^\dagger) &= A^\dagger \otimes A \\
\Psi(A \otimes N) &= N \otimes A \\
\Psi(A \otimes A) &= A \otimes A
\end{aligned} \tag{89}$$

where $[H] = \frac{1}{\omega}(e^{\frac{\omega H}{2}} - e^{-\frac{\omega H}{2}})$.

5 The n-body System of Braided Harmonic Oscillators

In the Introduction to this paper, it was mentioned that $U_q(h)$ was essentially the ‘quantum group version’ of the ordinary quantum harmonic oscillator. $BU_q(h)$, then, can similarly be considered as describing a braided harmonic oscillator. These oscillators both have the same algebra: the difference between them is that the latter is to be viewed as existing within a braided category, in the sense that was explained in the last section. This difference only becomes apparent when we consider a system of more than one oscillator. A system of n braided oscillators will be represented on the n -fold (braided) tensor product algebra [5, 6]. This algebra is as defined in Equation (2), iterated n times. We denote it $BU_q(h)^{\underline{\otimes} n}$. (It will be convenient, throughout this section, to distinguish between braided and unbraided tensor products by using an underlined symbol for the braided product.) In this Section, $BU_q(h)^{\underline{\otimes} n}$ and its representations will be studied with the above application in mind. We will attempt to define braided analogues of all the usual quantum-mechanical concepts: Fock space, an inner product, and time-evolution.

We begin by recalling that an algebra isomorphism exists between $U_q(h)$ and $BU_q(h)$, so that for a single oscillator, the braided and unbraided systems are equivalent. However, the braiding

would *a-priori* appear to cause significant differences in the tensor product algebras $U_q(h)^{\otimes n}$ and $BU_q(h)^{\underline{\otimes} n}$. The braided and unbraided n-body systems would therefore appear quite different, despite the identical nature of the individual oscillators which comprise them. The algebra of $U_q(h)^{\otimes n}$ is just the usual n-body harmonic oscillator algebra, with quantum group modification,

$$[A_i, A_j] = 0 \quad [A_i^\dagger, A_j^\dagger] = 0 \quad [A_i, A_j^\dagger] = \delta_{ij}[H_i] \quad (90)$$

$$[N_i, A_j] = -\delta_{ij}A_i \quad [N_i, A_j^\dagger] = \delta_{ij}A_j^\dagger \quad [N_i, N_j] = 0 \quad H_i \text{ central} \quad (91)$$

By contrast, the algebra of $BU_q(h)^{\underline{\otimes} n}$ is much more complicated, due to the braiding. It is calculated by repeated use of Equation (2), with Ψ as given in (89), to give

$$\begin{aligned} [\bar{A}_i, \bar{A}_i] &= 0 \\ [\bar{A}_i^\dagger, \bar{A}_j^\dagger] &= 0 \\ [\bar{A}_i, \bar{A}_j^\dagger] &= \begin{cases} \omega e^{-\frac{\omega}{4}\bar{H}_i} [\bar{H}_i] e^{-\frac{\omega}{4}\bar{H}_j} [\bar{H}_j] & i < j \\ [\bar{H}_i] & i = j \\ 0 & i > j \end{cases} \\ [\bar{N}_i, \bar{A}_j] &= \begin{cases} 0 & i < j \\ -\bar{A}_i & i = j \\ \omega e^{-\frac{\omega}{4}\bar{H}_j} [\bar{H}_j] e^{-\frac{\omega}{4}\bar{H}_i} \bar{A}_i & i > j \end{cases} \\ [\bar{N}_i, \bar{A}_j^\dagger] &= \begin{cases} \omega e^{-\frac{\omega}{4}\bar{H}_i} \bar{A}_i^\dagger e^{-\frac{\omega}{4}\bar{H}_j} [\bar{H}_j] & i < j \\ \bar{A}_i^\dagger & i = j \\ 0 & i > j \end{cases} \\ [\bar{N}_i, \bar{N}_j] &= \begin{cases} \omega e^{-\frac{\omega}{4}\bar{H}_i} \bar{A}_i^\dagger e^{-\frac{\omega}{4}\bar{H}_j} \bar{A}_j & i < j \\ 0 & i = j \end{cases} \\ [\bar{H}_i, \bar{Y}_j] &= 0 \quad \forall \bar{Y} = \bar{A}, \bar{A}^\dagger, \bar{N}, \bar{H} \end{aligned} \quad (92)$$

where A_i (\bar{A}_i) is that element of $U_q(h)^{\otimes n}$ ($BU_q(h)^{\underline{\otimes} n}$) which has A in the i^{th} place and 1 elsewhere; and equivalently for the other generators. This definition will hold throughout this section, except where otherwise explicitly noted. Elements of $BU_q(h)^{\underline{\otimes} n}$ are barred to make explicit the distinction between braided and unbraided tensor products. The notation

$$[H] = \frac{1}{\omega} (e^{\frac{\omega H}{2}} - e^{-\frac{\omega H}{2}}) \quad (93)$$

will also be adopted universally.

Note that the *order of labelling* matters in $BU_q(h)^{\otimes n}$. This, though a natural consequence of the braiding, would seem a little odd physically: exchanging the labels of two oscillators changes the *value* (and not just the sign) of some of their commutators. However, this peculiarity of braided statistics may not actually be manifest in physical states of the system, due to the existence of an isomorphism, (which will be demonstrated later this section), between braided and unbraided tensor products.

Having written down the n-fold braided tensor product algebra, the next task is to find a suitable Fock Space representation for it. Such representations of $U_q(h)$ are already known: a generic definition is given in [3]

$$\begin{aligned} A|r\rangle &= [\hbar]^{\frac{1}{2}}\sqrt{r}|r-1\rangle \\ A^\dagger|r\rangle &= [\hbar]^{\frac{1}{2}}\sqrt{r+1}|r+1\rangle \\ H|r\rangle &= \hbar|r\rangle \\ N|r\rangle &= (n' + r)|r\rangle \end{aligned} \tag{94}$$

where $\{|r\rangle\}_{r=0}^\infty$ is an orthonormal basis; and the irrep is labelled by the eigenvalues of the central elements H and $C(= [H]N - A^\dagger A)$ (being \hbar and $[\hbar]n'$ respectively). It is obvious from the above definition that any state $|r\rangle$ can be expressed in terms of the vacuum as follows

$$|r\rangle = \frac{(A^\dagger)^r}{[\hbar]^{\frac{r}{2}}\sqrt{r!}}|0\rangle \tag{95}$$

and that therefore the given irreps are indeed Fock Space representations. It is quite easy to generalise the representation from 1 to n oscillators for $U_q(h)$. A tensor product of the single oscillator bases will form a Fock Space representation of $U_q(h)^{\otimes n}$.

$$\begin{aligned} |r_1, r_2, \dots, r_n\rangle &= |r_1\rangle \otimes |r_2\rangle \otimes \dots \otimes |r_n\rangle \\ &= \frac{(A_1^\dagger)^{r_1}(A_2^\dagger)^{r_2} \dots (A_n^\dagger)^{r_n}}{[\hbar_1]^{\frac{r_1}{2}} \dots [\hbar_n]^{\frac{r_n}{2}} \sqrt{r_1! r_2! \dots r_n!}} |0\rangle \otimes |0\rangle \otimes \dots \otimes |0\rangle \end{aligned} \tag{96}$$

Note that the individual oscillators need not all be represented by the same irrep. To maintain generality, the eigenvalues of H_i and C_i in the i^{th} oscillator are denoted by \hbar_i and $[\hbar_i]n'_i$ respectively. “Ordinary” quantum mechanics can be obtained from this more general ‘quantum group

version', by setting $\hbar_i = \hbar$ and $n'_i = 0, \forall i$. The system therefore consists of n independent harmonic oscillators, each of which has fixed, though possibly different, values of Planck's constant and the vacuum expectation value of the number operator.

For the braided algebra, the situation is not so trivial. At the single oscillator level, $BU_q(h)$ obviously has the same representations as $U_q(h)$, since it has the same algebra. An n -fold basis can be constructed in an analogous way to that for $U_q(h)^{\otimes n}$, by taking now the *braided* tensor product of the individual bases. Thus

$$|r_1, \dots, r_n\rangle = |r_1\rangle \underline{\otimes} |r_2\rangle \underline{\otimes} \dots \underline{\otimes} |r_n\rangle \quad (97)$$

However, this cannot be written as a Fock Space in any obvious manner, since because of the braiding the A^\dagger 's will not necessarily commute with the states. To find the action of any of the operators on the states as defined above, we must first compute the braiding between the generators and the (single-oscillator) states.

The braiding can be calculated from the universal R -matrix, given in Equation (55), and the known actions of $U_q(h)$ on both $BU_q(h)$ and the representation (Equations (88) and (94) respectively), using the standard formula [8, section 7]:

$$\Psi(v \underline{\otimes} w) = \sum \mathcal{R}^{(2)} \triangleright w \underline{\otimes} \mathcal{R}^{(1)} \triangleright v \quad (\mathcal{R} = \sum \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}) \quad (98)$$

The results are as follows

$$\begin{aligned} \Psi(A \underline{\otimes} |r_i\rangle) &= e^{\frac{\omega}{2}\hbar_i} |r_i\rangle \underline{\otimes} A \\ \Psi(A^\dagger \underline{\otimes} |r_i\rangle) &= \omega e^{-\frac{\omega}{2}\hbar_i} [\hbar_i]^{\frac{1}{2}} \sqrt{r_i + 1} |r_i + 1\rangle \underline{\otimes} e^{-\frac{\omega}{4}H}[H] + e^{-\frac{\omega}{2}\hbar_i} |r_i\rangle \underline{\otimes} A^\dagger \\ \Psi(N \underline{\otimes} |r_i\rangle) &= \omega e^{\frac{\omega}{4}\hbar_i} [\hbar_i]^{\frac{1}{2}} \sqrt{r_i + 1} |r_i + 1\rangle \underline{\otimes} e^{-\frac{\omega}{4}H}A + |r_i\rangle \underline{\otimes} N \\ \Psi(H \underline{\otimes} |r_i\rangle) &= |r_i\rangle \underline{\otimes} H \end{aligned} \quad (99)$$

We are now in a position to calculate the action of arbitrary tensor products of operators on the braided space. To calculate such an action, each operator must be braided through all the states until it reaches the one on which it should act; only when this has been done for all

operators should the actions be computed. Proceeding in this fashion, we find the action of A_i^\dagger on a general n -fold state to be

$$\begin{aligned} \bar{A}_i^\dagger |r_1, \dots, r_n\rangle &= \omega e^{-\frac{\omega}{4}\hbar_i} [\hbar_i] \sum_{j=1}^{i-1} \{e^{-\frac{\omega}{2}(\hbar_1+\dots+\hbar_{j-1})} e^{-\frac{\omega}{4}\hbar_j} [\hbar_j]^{\frac{1}{2}} \sqrt{r_j+1} |r_1, \dots, r_j+1, \dots, r_n\rangle\} \\ &\quad + e^{-\frac{\omega}{2}(\hbar_1+\dots+\hbar_{i-1})} [\hbar_i]^{\frac{1}{2}} \sqrt{r_i+1} |r_1, \dots, r_i+1, \dots, r_n\rangle \end{aligned} \quad (100)$$

If we postulate the existence of some operators \bar{X}_i^\dagger which have the following action on a general state

$$\bar{X}_i^\dagger |r_1, \dots, r_n\rangle = [\hbar_i]^{\frac{1}{2}} \sqrt{r_i+1} |r_1, \dots, r_i+1, \dots, r_n\rangle \quad (101)$$

then it is clear that \bar{A}_i^\dagger can be written as a sum of such operators

$$A_i^\dagger = e^{-\frac{\omega}{2}(\bar{H}_1+\dots+\bar{H}_{i-1})} \bar{X}_i^\dagger + \omega e^{-\frac{\omega}{4}\bar{H}_i} [\bar{H}_i] \sum_{j=1}^{i-1} \{e^{-\frac{\omega}{2}(\bar{H}_1+\dots+\bar{H}_{j-1})} e^{-\frac{\omega}{4}\bar{H}_j} \bar{X}_j^\dagger\} \quad (102)$$

This relation can then be inverted to give a definition of \bar{X}_i^\dagger in terms of the \bar{A}_i^\dagger 's

$$\bar{X}_i^\dagger = e^{\frac{\omega}{2}(\bar{H}_1+\dots+\bar{H}_{i-1})} \bar{A}_i^\dagger - \omega [\bar{H}_i] e^{-\frac{\omega}{4}\bar{H}_i} \sum_{j=1}^{i-1} \{e^{\frac{\omega}{2}(\bar{H}_1+\dots+\bar{H}_{j-1})} e^{\frac{\omega}{4}\bar{H}_j} e^{-\frac{\omega}{2}(\bar{H}_{j+1}+\dots+\bar{H}_{i-1})} \bar{A}_j^\dagger\} \quad (103)$$

It is clear from this last equation that the \bar{X}^\dagger 's are properly defined operators in $BU_q(h)^{\otimes n}$, although the subscript notation must be understood differently from the usual convention. Written out as a tensor product, \bar{X}_i^\dagger does *not* consist of \bar{X}^\dagger in the i^{th} place and 1 elsewhere; it is a sum of such elements, and will contain elements different from 1 in all places up to and including the i^{th} .

In effect, \bar{X}_i^\dagger can be considered as a ‘diagonalised version’ of \bar{A}_i^\dagger . It is clear from the action of the \bar{X}^\dagger 's on the representation that it is they, *and not the A^\dagger 's*, which are the physical creation operators on the braided space. The existence of this diagonalisation implies that the braided representation of $BU_q(h)^{\otimes n}$ (Equation (97)) is a Fock Space; explicitly

$$|r_1, \dots, r_n\rangle = \frac{(\bar{X}_1^\dagger)^{r_1} \dots (\bar{X}_n^\dagger)^{r_n}}{[\hbar_1]^{\frac{r_1}{2}} \dots [\hbar_n]^{\frac{r_n}{2}} \sqrt{r_1! \dots r_n!}} |0\rangle \underline{\otimes} \dots \underline{\otimes} |0\rangle \quad (104)$$

The similarity of this formula to Equation (96) suggests the possible existence of an isomorphism between the two Fock Spaces.

Theorem 5.1 *The algebras $BU_q(h)^{\otimes n}$ and $U_q(h)^{\otimes n}$ are isomorphic.*

Proof We begin by ‘diagonalising’ \bar{A} , as it is the diagonalised versions of \bar{A}^\dagger and \bar{A} which we suspect (due to the similarity of formulas (96) and (104)) may be isomorphic to the unbraided tensor algebra. The action of \bar{A} on a general state is

$$\bar{A}_i |r_1, \dots, r_n\rangle = e^{\frac{\omega}{2}(\hbar_1 + \dots + \hbar_{i-1})} [\hbar_i]^{\frac{1}{2}} \sqrt{r_i} |r_1, \dots, r_i - 1, \dots, r_n\rangle \quad (105)$$

From this action it is clear that if \bar{X}_i is defined as follows in terms of \bar{A}_i

$$\bar{X}_i = e^{-\frac{\omega}{2}(\bar{H}_1 + \dots + \bar{H}_{i-1})} \bar{A}_i \quad (106)$$

then it will have the desired diagonal action on a general state

$$\bar{X}_i |r_1, \dots, r_n\rangle = [\hbar_i]^{\frac{1}{2}} \sqrt{r_i} |r_1, \dots, r_i - 1, \dots, r_n\rangle \quad (107)$$

We now proceed to compute the commutation relations of \bar{X}_i^\dagger and \bar{X}_i , as determined by the braided relations between \bar{A}_i^\dagger and \bar{A}_i , given in Equation (92). Since \bar{X}_i^\dagger is a function only of \bar{A}_i^\dagger , we have immediately

$$[\bar{X}_i^\dagger, \bar{X}_j^\dagger] = 0 \quad (108)$$

and similarly, noting that \bar{X}_i likewise is a function only of \bar{A}_i

$$[\bar{X}_i, \bar{X}_j] = 0 \quad (109)$$

Since the commutator $[\bar{A}_i, \bar{A}_j^\dagger]$ has different values depending on the order of i and j, it will be convenient to calculate the three cases for $[\bar{X}_i, \bar{X}_j^\dagger]$ separately.

$$\begin{aligned} [\bar{X}_i, \bar{X}_j^\dagger]_{i>j} &= [e^{-\frac{\omega}{2}(\bar{H}_1 + \dots + \bar{H}_{i-1})} \bar{A}_i, e^{\frac{\omega}{2}(\bar{H}_1 + \dots + \bar{H}_{j-1})} \bar{A}_j^\dagger] \\ &\quad - \omega [\bar{H}_j] e^{-\frac{\omega}{4}\bar{H}_j} \sum_{k=1}^{j-1} \{ e^{\frac{\omega}{2}(\bar{H}_1 + \dots + \bar{H}_{k-1})} e^{\frac{\omega}{4}\bar{H}_k} e^{-\frac{\omega}{2}(\bar{H}_{k+1} + \dots + \bar{H}_{j-1})} \bar{A}_j^\dagger \} \\ &= 0 \end{aligned} \quad (110)$$

$$\begin{aligned} [\bar{X}_i, \bar{X}_i^\dagger] &= [e^{-\frac{\omega}{2}(\bar{H}_1 + \dots + \bar{H}_{i-1})} \bar{A}_i, e^{\frac{\omega}{2}(\bar{H}_1 + \dots + \bar{H}_{i-1})} \bar{A}_i^\dagger] \\ &\quad - \omega [\bar{H}_i] e^{-\frac{\omega}{4}\bar{H}_i} \sum_{j=1}^{i-1} \{ e^{\frac{\omega}{2}(\bar{H}_1 + \dots + \bar{H}_{j-1})} e^{\frac{\omega}{4}\bar{H}_j} e^{-\frac{\omega}{2}(\bar{H}_{j+1} + \dots + \bar{H}_{i-1})} \bar{A}_i^\dagger \} \end{aligned}$$

$$\begin{aligned}
&= [\bar{A}_i, \bar{A}_i^\dagger] \\
&= [\bar{H}_i]
\end{aligned} \tag{111}$$

$$\begin{aligned}
[\bar{X}_i, \bar{X}_j^\dagger]_{i < j} &= [e^{-\frac{\omega}{2}(\bar{H}_1 + \dots + \bar{H}_{i-1})} \bar{A}_i, e^{\frac{\omega}{2}(\bar{H}_1 + \dots + \bar{H}_{j-1})} \bar{A}_j^\dagger \\
&\quad - \omega [\bar{H}_j] e^{-\frac{\omega}{4}\bar{H}_j} \sum_{k=1}^{j-1} \{e^{\frac{\omega}{2}(\bar{H}_1 + \dots + \bar{H}_{k-1})} e^{\frac{\omega}{4}\bar{H}_k} e^{-\frac{\omega}{2}(\bar{H}_{k+1} + \dots + \bar{H}_{j-1})} \bar{A}_k^\dagger\}] \\
&= e^{-\frac{\omega}{2}(\bar{H}_1 + \dots + \bar{H}_{i-1})} e^{\frac{\omega}{2}(\bar{H}_1 + \dots + \bar{H}_{i-1})} [\bar{A}_i, e^{\frac{\omega}{2}(\bar{H}_i + \dots + \bar{H}_{j-1})} \bar{A}_j^\dagger \\
&\quad - \omega [\bar{H}_j] e^{-\frac{\omega}{4}\bar{H}_j} \sum_{k=i}^{j-1} \{e^{\frac{\omega}{2}(\bar{H}_i + \dots + \bar{H}_{k-1})} e^{\frac{\omega}{4}\bar{H}_k} e^{-\frac{\omega}{2}(\bar{H}_{k+1} + \dots + \bar{H}_{j-1})} \bar{A}_k^\dagger\}] \\
&= e^{\frac{\omega}{2}(\bar{H}_i + \dots + \bar{H}_{j-1})} [\bar{A}_i, \bar{A}_j^\dagger] - \omega [\bar{H}_j] e^{-\frac{\omega}{4}\bar{H}_j} e^{\frac{\omega}{4}\bar{H}_i} e^{-\frac{\omega}{2}(\bar{H}_{i+1} + \dots + \bar{H}_{j-1})} [\bar{A}_i, \bar{A}_i^\dagger] \\
&\quad - \omega [\bar{H}_j] e^{-\frac{\omega}{4}\bar{H}_j} \sum_{k=i+1}^{j-1} \{e^{\frac{\omega}{2}(\bar{H}_i + \dots + \bar{H}_{k-1})} e^{\frac{\omega}{4}\bar{H}_k} e^{-\frac{\omega}{2}(\bar{H}_{k+1} + \dots + \bar{H}_{j-1})} [\bar{A}_i, \bar{A}_k^\dagger]\} \\
&= e^{\frac{\omega}{2}(\bar{H}_i + \dots + \bar{H}_{j-1})} \omega e^{-\frac{\omega}{4}\bar{H}_i} [\bar{H}_i] e^{-\frac{\omega}{4}\bar{H}_j} [\bar{H}_j] \\
&\quad - \omega [\bar{H}_j] e^{-\frac{\omega}{4}\bar{H}_j} e^{\frac{\omega}{4}\bar{H}_i} e^{-\frac{\omega}{2}(\bar{H}_{i+1} + \dots + \bar{H}_{j-1})} [\bar{H}_i] \\
&\quad - \omega [\bar{H}_j] e^{-\frac{\omega}{4}\bar{H}_j} \sum_{k=i+1}^{j-1} \{e^{\frac{\omega}{2}(\bar{H}_i + \dots + \bar{H}_{k-1})} \\
&\quad e^{\frac{\omega}{4}\bar{H}_k} e^{-\frac{\omega}{2}(\bar{H}_{k+1} + \dots + \bar{H}_{j-1})} \omega e^{-\frac{\omega}{4}\bar{H}_i} [\bar{H}_i] e^{-\frac{\omega}{4}\bar{H}_k} [\bar{H}_k]\} \\
&= \omega e^{-\frac{\omega}{4}(\bar{H}_i + \bar{H}_j)} [\bar{H}_i][\bar{H}_j] \{e^{\frac{\omega}{2}(\bar{H}_i + \dots + \bar{H}_{j-1})} - e^{\frac{\omega}{2}\bar{H}_i} e^{-\frac{\omega}{2}(\bar{H}_{i+1} + \dots + \bar{H}_{j-1})} \\
&\quad - \sum_{k=i+1}^{j-1} \{e^{\frac{\omega}{2}(\bar{H}_i + \dots + \bar{H}_{k-1})} e^{-\frac{\omega}{2}(\bar{H}_{k+1} + \dots + \bar{H}_{j-1})} (e^{\frac{\omega}{2}\bar{H}_k} - e^{-\frac{\omega}{2}\bar{H}_k})\}\} \\
&= \omega e^{-\frac{\omega}{4}(\bar{H}_i + \bar{H}_j)} [\bar{H}_i][\bar{H}_j] \{e^{\frac{\omega}{2}(\bar{H}_i + \dots + \bar{H}_{j-1})} - e^{\frac{\omega}{2}\bar{H}_i} e^{-\frac{\omega}{2}(\bar{H}_{i+1} + \dots + \bar{H}_{j-1})} \\
&\quad - \sum_{k=i+1}^{j-1} \{e^{\frac{\omega}{2}(\bar{H}_i + \dots + \bar{H}_k)} e^{-\frac{\omega}{2}(\bar{H}_{k+1} + \dots + \bar{H}_{j-1})}\} \\
&\quad + \sum_{k=i+1}^{j-1} \{e^{\frac{\omega}{2}(\bar{H}_i + \dots + \bar{H}_{k-1})} e^{-\frac{\omega}{2}(\bar{H}_k + \dots + \bar{H}_{j-1})}\}\} \\
&= \omega e^{-\frac{\omega}{4}(\bar{H}_i + \bar{H}_j)} [\bar{H}_i][\bar{H}_j] \{e^{\frac{\omega}{2}(\bar{H}_i + \dots + \bar{H}_{j-1})} - e^{\frac{\omega}{2}\bar{H}_i} e^{-\frac{\omega}{2}(\bar{H}_{i+1} + \dots + \bar{H}_{j-1})} \\
&\quad - e^{\frac{\omega}{2}(\bar{H}_i + \dots + \bar{H}_{j-1})} + e^{\frac{\omega}{2}\bar{H}_i} e^{-\frac{\omega}{2}(\bar{H}_{i+1} + \dots + \bar{H}_{j-1})}\} v \\
&= 0
\end{aligned} \tag{112}$$

In Equation (112), the second-last equality is obtained by writing out both sums; all but one term in each cancel pairwise with terms in the other. So, combining the above three equations,

we have

$$[\bar{X}_i, \bar{X}_j^\dagger] = \delta_{ij}[\bar{H}_i] \quad (113)$$

which is the same as the *unbraided* commutator between A_i and A_j^\dagger , assuming that \bar{H}_i is identified with H_i . This is certainly possible, as the central elements of two algebras can always be identified.

By the same argument, the quadratic casimirs C_i and \bar{C}_i can also be identified, and from this the element (\bar{M}_i) of $BU_q(h)^{\otimes n}$ which is isomorphic to N_i can be found. It is

$$\bar{M}_i = \bar{N}_i + \frac{1}{[\bar{H}_i]} \{ \bar{X}_i^\dagger \bar{X}_i - \bar{A}_i^\dagger \bar{A}_i \} \quad (= \frac{1}{[\bar{H}_i]} \{ \bar{C}_i + \bar{X}_i^\dagger \bar{X}_i \}) \quad (114)$$

It is easily seen (by inspection) that \bar{M}_i will act on the representation as a ‘diagonalised version’ of \bar{N}_i

$$\bar{M}_i |r_1, \dots, r_n\rangle = (n'_i + r_i) |r_1, \dots, r_n\rangle \quad (115)$$

Thus the identifications

$$\begin{aligned} \bar{X}_i^\dagger &= A_i^\dagger \\ \bar{X}_i &= A_i \\ \bar{H}_i &= H_i \\ \bar{M}_i &= N_i \end{aligned} \quad (116)$$

define an isomorphism between $BU_q(h)^{\otimes n}$ and $U_q(h)^{\otimes n}$. \square

The form of the isomorphism makes it clear that the braided and unbraided Fock Spaces are indeed equivalent. The strange dependence of the braided generators on the order of labeling *does not appear* in the physical creation and annihilation operators, \bar{X}_i and \bar{X}_i^\dagger . Physical states of the system will therefore be unaffected by the braided statistics.

Having found a Fock Space representation for $BU_q(h)^{\otimes n}$, we now turn to the question of whether a $*$ -structure can be found which will allow an inner product to be defined between the Fock Space states.

Corollary 5.2 *A $*$ -structure exists on $BU_q(h)^{\otimes n}$, which is consistent with orthonormality of the Fock Space basis.*

Proof We begin by considering the 1-dimensional problem. We want our representation to form an orthonormal basis for the Fock Space, so we define the inner product

$$(|n\rangle, |m\rangle) = \delta_{n,m} \quad (117)$$

This then defines the $*$ -structure of the generators, through the property

$$(|n\rangle, a|m\rangle) = (a^*|n\rangle, |m\rangle) \quad (118)$$

It can immediately be seen that the $*$ -structure

$$(A^\dagger)^* = A, \quad A^* = A^\dagger, \quad N^* = N, \quad H^* = H \quad (119)$$

is consistent with the inner product as defined. This is a known $*$ -structure of $U_q(h)$: it can easily be checked that $*$ as defined in Equation (119) extends as an anti-algebra map, if ω is assumed to be real.

This $*$ -structure must now be extended to encompass the n -dimensional system. This can easily be done for $U_q(h)^{\otimes n}$. Clearly the inner product

$$(|r_1, \dots, r_n\rangle, |s_1, \dots, s_n\rangle) = \delta_{r_1, s_1} \delta_{r_2, s_2} \cdots \delta_{r_n, s_n} \quad (120)$$

immediately implies

$$A_i^* = A_i^\dagger, \quad (A_i^\dagger)^* = A_i, \quad N_i^* = N_i, \quad H_i^* = H_i \quad (121)$$

by exactly the same argument as was used for the 1-dimensional problem. It is also immediately clear that $*$ extends as an anti-algebra map for these operators, and therefore for all elements of $U_q(h)^{\otimes n}$.

The isomorphism can now be invoked to transfer this result directly to $BU_q(h)^{\otimes n}$. The inner product as defined in Equation (120) will be consistent with the following $*$ -structure on the

‘diagonalised’ generators

$$\bar{X}_i^* = \bar{X}_i^\dagger, \quad (\bar{X}_i^\dagger)^* = \bar{X}_i, \quad \bar{M}_i^* = \bar{M}_i, \quad \bar{H}_i^* = \bar{H}_i \quad (122)$$

This then implies a rather complicated $*$ -structure for the usual generators

$$\begin{aligned} \bar{A}_i^* &= e^{\omega(\bar{H}_1 + \dots + \bar{H}_{i-1})} \bar{A}_i^\dagger - \omega [\bar{H}_i] e^{-\frac{\omega}{4} \bar{H}_i} \sum_{j=1}^{i-1} \{ e^{\omega(\bar{H}_1 + \dots + \bar{H}_{j-1})} e^{\frac{3\omega}{4} \bar{H}_j} \bar{A}_j^\dagger \} \\ (\bar{A}_i^\dagger)^* &= e^{-\omega(\bar{H}_1 + \dots + \bar{H}_{i-1})} \bar{A}_i + \omega [\bar{H}_i] e^{-\frac{\omega}{4} \bar{H}_i} \sum_{j=1}^{i-1} \{ e^{-\omega(\bar{H}_1 + \dots + \bar{H}_{j-1})} e^{-\frac{\omega}{4} \bar{H}_j} \bar{A}_j \} \\ \bar{N}_i^* &= \bar{N}_i - \omega \sum_{j=1}^{i-1} (e^{-\omega(\bar{H}_{j+1} + \dots + \bar{H}_{i-1})} e^{-\frac{\omega}{4}(\bar{H}_i + \bar{H}_j)} \bar{A}_i^\dagger \bar{A}_j) \\ &\quad + \omega \sum_{j=1}^{i-1} (e^{\omega(\bar{H}_{j+1} + \dots + \bar{H}_{j-1})} e^{-\frac{\omega}{4} \bar{H}_i} e^{\frac{3\omega}{4} \bar{H}_j} \bar{A}_i^\dagger \bar{A}_j) \\ &\quad - \omega^2 [\bar{H}_i] e^{-\frac{\omega}{2} \bar{H}_i} \left(\sum_{j=1}^{i-1} e^{\omega(\bar{H}_1 + \dots + \bar{H}_{j-1})} e^{\frac{3\omega}{4} \bar{H}_j} \bar{A}_j^\dagger \right) \left(\sum_{k=1}^{i-1} e^{-\omega(\bar{H}_1 + \dots + \bar{H}_{k-1})} e^{-\frac{\omega}{4} \bar{H}_k} \bar{A}_k \right) \end{aligned} \quad (123)$$

□

The last concept needed in order to fully define the quantum mechanics of braided harmonic oscillators is their time evolution. It would be nice to be able to describe this time evolution in terms of an action of $U_q(h)$ on $BU_q(h)$, since this would ensure that all braided group maps were preserved. We therefore search for an element of $U_q(h)$ whose action on $BU_q(h)$ produces a physically reasonable time evolution.

An obvious choice is the quadratic casimir: $C = [H]N - A^\dagger A$, since being central it will preserve the symmetry of the system during evolution. Its actions on the generators of $BU_q(h)$ are as follows

$$C \triangleright \begin{pmatrix} A^\dagger & N \\ H & A \end{pmatrix} = \begin{pmatrix} 0 & e^{-\frac{\omega}{2} H} [H] \\ 0 & 0 \end{pmatrix} \quad (124)$$

This does not provide a very physical time evolution, since if A and A^\dagger do not evolve in time, neither will the states of the system. Only N evolves, and this is an operator which should *not* evolve in time, as this would imply change in the irrep describing the system.

Discarding the casimir as an option, then, another possibility is to choose a primitive element of $U_q(h)$. This would have the advantage of giving equations of motion very similar to the

Heisenberg picture. Since H has a zero action on all elements, we shall consider the action of N . It is, as was given in Equation (88)

$$N \triangleright \begin{pmatrix} A^\dagger & N \\ H & A \end{pmatrix} = \begin{pmatrix} A^\dagger & 0 \\ 0 & -A \end{pmatrix} \quad (125)$$

This makes much more sense in terms of the representation: it leaves alone those operators whose values are constant in a given irrep; and evolves those which evolve the states. To turn this action explicitly into time evolution, we can look at the action of $e^{\frac{itN}{\hbar}}$. This gives the same time evolution as we would derive from the Heisenberg picture taking N as the Hamiltonian. For example

$$\dot{A}^\dagger = \frac{i}{\hbar}[N, A^\dagger] = \frac{i}{\hbar}A^\dagger \quad (126)$$

$$\implies A^\dagger(t) = e^{\frac{it}{\hbar}N} A^\dagger (= e^{\frac{itN}{\hbar}} \triangleright A^\dagger) \quad (127)$$

and similarly for A .

Although finding a Hamiltonian for the n-body braided system might seem highly non-trivial, the time evolution as expressed by the action *can* easily be extended to n dimensions. An action on a tensor product is defined by [5]

$$h \triangleright (v \otimes w) = h_{(1)} \triangleright v \otimes h_{(2)} \triangleright w \quad \Delta h = h_{(1)} \otimes h_{(2)} \quad (128)$$

Since

$$Y \triangleright 1 = 0 \quad \forall Y = A, A^\dagger, H, N \quad (129)$$

only one term in the coproduct of any of the generators contributes to the action (128). This action is exactly the same as that given in Equation (88), with the addition of the subscript i throughout. It is then obvious that the action of e^{itN} , our time evolution, is as follows

$$e^{itN} \triangleright \begin{pmatrix} \bar{A}_i^\dagger & \bar{N}_i \\ \bar{H}_i & \bar{A}_i \end{pmatrix} = \begin{pmatrix} e^{it} \bar{A}_i^\dagger & \bar{N}_i \\ \bar{H}_i & e^{-it} \bar{A}_i \end{pmatrix} \quad (130)$$

We omit the \hbar here, since it is now somewhat ambiguous as to how it should be included, when it can take a different value for each oscillator. It then follows, since \bar{X}_i^\dagger is just a sum of A_i^\dagger 's

(and similarly for \bar{X}_i), that the time evolution of these ‘diagonal generators’ is

$$e^{itN} \triangleright \bar{X}_i^\dagger = e^{it} \bar{X}_i^\dagger, \quad e^{itN} \triangleright \bar{X}_i = e^{-it} \bar{X}_i \quad (131)$$

Using these results, the time evolution of \bar{M}_i can also be calculated

$$e^{itN} \triangleright \bar{M}_i = \bar{M}_i \quad (132)$$

We can then use the isomorphism in Theorem 4.3 to find the corresponding time evolution of the generators of $U_q(h)^{\otimes n}$

$$e^{itN} \triangleright A_i^\dagger = e^{it} A_i^\dagger, \quad e^{itN} \triangleright A_i = e^{-it} A_i, \quad e^{itN} \triangleright H_i = H_i, \quad e^{itN} \triangleright N_i = N_i \quad (133)$$

It is then fairly easy, due to the relative simplicity of the unbraided algebra, to find an element of $U_q(h)^{\otimes n}$ which will act as a Hamiltonian. We need an element \mathcal{H} which has the following commutators with A_i^\dagger , A_i and H_i

$$[\mathcal{H}, A_i^\dagger] = A_i^\dagger, \quad [\mathcal{H}, A_i] = -A_i, \quad [\mathcal{H}, H_i] = 0 \quad (134)$$

The element

$$\mathcal{H} = \sum_i N_i \quad (135)$$

obviously meets these requirements, and is therefore a suitable Hamiltonian. There is a very close analogy between this expression and the usual quantum-mechanical solution for a system of non-interacting harmonic oscillators. The action we have chosen produces the usual time evolution for the free system!

Having found the Hamiltonian for the unbraided algebra, it can now be mapped back to the braided algebra via the isomorphism,

$$\bar{\mathcal{H}} = \sum_i \bar{M}_i = \sum_i \{ \bar{N}_i - \omega e^{-\frac{\omega}{4} \bar{H}_i} \sum_{j=1}^{i-1} (e^{-\frac{\omega}{4} \bar{H}_j} e^{-\omega(\bar{H}_{j+1} + \dots + \bar{H}_{i-1})} \bar{A}_j^\dagger \bar{A}_i) \} \quad (136)$$

The action of the other generators of $U_q(h)$ on the ‘diagonal generators’ of $BU_q(h)^{\otimes n}$, is only slightly more complicated to calculate than that of N , following a similar method. The

results can then be mapped across to $U_q(h)^{\otimes n}$ in exactly the same way as was done for the time evolution. H of course has a zero action as always. The actions of the other two generators are

$$\begin{aligned}
A \triangleright \begin{pmatrix} A_i^\dagger & N_i \\ H_i & A_i \end{pmatrix} &= \begin{pmatrix} e^{-\frac{\omega}{2}(H_1+\dots+H_{i-1})} e^{-\frac{\omega H_i}{4}} [H_i] & e^{-\frac{\omega}{2}(H_1+\dots+H_{i-1})} e^{-\frac{\omega H_i}{4}} A_i \\ 0 & 0 \end{pmatrix} \\
A^\dagger \triangleright \begin{pmatrix} A_i^\dagger & N_i \\ H_i & A_i \end{pmatrix} &= \begin{pmatrix} 0 & -e^{-\frac{\omega}{2}(H_1+\dots+H_{i-1})} e^{-\frac{\omega H_i}{4}} A_i^\dagger \\ 0 & -e^{-\frac{\omega}{2}(H_1+\dots+H_{i-1})} e^{-\frac{\omega H_i}{4}} [H_i] \end{pmatrix}
\end{aligned} \tag{137}$$

In summary, the isomorphism between the braided and unbraided tensor product algebras reveals a hitherto unsuspected structure underlying the ‘normal’ unbraided system. The braided tensor product $BU_q(h)^{\otimes n}$ is $U_q(h)$ -covariant by construction: the isomorphism demonstrates that $U_q(h)^{\otimes n}$ shares this property. Furthermore, the action of N on $U_q(h)^{\otimes n}$ which is deduced in this way turns out to be the free particle evolution. Thus the time evolution is identified as part of a quantum group symmetry of the n -fold harmonic oscillator. While these results are not obvious from the perspective of the familiar, unbraided system, we note that on the braided side they follow automatically from standard constructions. It is to be expected that other constructions which are quite natural from the braided group point of view, may similarly have important implications for the physical system.

References

- [1] E.Celeghini, R.Giachetti, E.Sorace and M.Tarlini, The Quantum Heisenberg Group. $H(1)_q$. *J. Math. Phys.*, **32** (1991) 1155.
- [2] Drinfeld “Quantum Groups”. In A. Gleason (ed), *Proceedings of the ICM, Amer. Math. Soc.*, Rhode Island (1987) 798.
- [3] C.Gómez and G.Sierra, Quantum Harmonic Oscillator Algebra and Link Invariants. *Preprint* (1991).
- [4] L.Faddeev, N.Yu.Reshetikin and L.Takhtajan, Quantisation of Lie Groups and Lie Algebras. *Alg. i. Analiz.*, **1(1)** (1989) 178.

- [5] S.Majid, Braided Groups and Algebraic Quantum Field Theories. *Lett. Math. Phys.*, **22** (1991) 167.
- [6] S.Majid, Examples of Braided Groups and Braided Matrices. *J. Math. Phys.*, **32** (1991) 3246.
- [7] S.Majid, Braided Matrix Structure of the Sklyanin Algebra and of the Quantum Lorentz Group. *Preprint* (1992).
- [8] S.Majid, Quasitriangular Hopf Algebras and Yang-Baxter Equations. *Int. J. Mod. Phys. A*, **5(1)** (1990) 1.